MEASURES OF WEAK NONCOMPACTNESS IN BANACH SPACES

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To the memory of our friend Jan Pelant

ABSTRACT. Measures of weak noncompactness are formulae that *quantify* different characterizations of weak compactness in Banach spaces: we deal here with De Blasi's measure ω and the measure of double limits γ inspired by Grothendieck's characterization of weak compactness. Moreover for bounded sets H of a Banach space E we consider the *worst* distance $\mathrm{k}(H)$ of the weak*-closure in the bidual $\overline{H}$ of H to E and the *worst* distance $\mathrm{ck}(H)$ of the sets of weak*-cluster points in the bidual of sequences in H to E. We prove the inequalities

$$\mathrm{ck}(H) \overset{\text{(I)}}{\leq} \mathrm{k}(H) \leq \gamma(H) \overset{\text{(II)}}{\leq} 2\,\mathrm{ck}(H) \leq 2\,\mathrm{k}(H) \leq 2\omega(H)$$

which say that ck, k and γ are equivalent. If E has Corson property $\mathcal{C}$ then (I) is always an equality but in general constant 2 in (II) is needed: we indeed provide an example for which $\mathrm{k}(H) = 2\,\mathrm{ck}(H)$. We obtain quantitative counterparts to Eberlein-Smulyan's and Gantmacher's theorems using γ. Since it is known that Gantmacher's theorem cannot be quantified using ω we therefore have another proof of the fact that γ and ω are not equivalent. We also offer a quantitative version of the classical Grothendieck's characterization of weak compactness in spaces $C(K)$ using γ.

1. INTRODUCTION

We use topological tools to study measures of weak noncompactness in Banach spaces. Measures of noncompactness or weak noncompactness have been successfully applied in operator theory, differential equations and integral equations, see for instance [1, 3, 4, 9, 14, 15] and [16]. We deal here with the following non-negative functions defined on the family of bounded sets H of Banach spaces E, see Definition 1:

- $\omega(H)$ is the worst distance from H to weakly compact sets of E,
- $\gamma(H)$ is the worst distance between iterated limits for sequences in H and sequences in the dual unit ball B_{E^*},
- $\mathrm{k}(H)$ is the worst distance to E of points of the weak*-closure $\overline{H}^{w^*}$ of H in the bidual E^{**},
- $\mathrm{ck}(H)$ is the *worst* distance to E of the sets of weak*-cluster points in the bidual E^{**} of sequences in H.

The function ω was introduced by de Blasi [4] as a measure of weak noncompactness that can be regarded as the counterpart for the weak topology of the classical Hausdorff measure of norm noncompactness. The function γ already appeared

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in [1] and in [15, Theorem 2.2]: in the latter the $\sup$ is taken over all the sequences in the convex hull $\operatorname{conv} H$ instead of sequences only in H: very recently γ has been implicitly used in [5] and [8] where it has been proved, amongst other things, that $\gamma(H) = \gamma(\operatorname{conv}(H))$ which says that our definition for γ is equivalent to the one given in [15]. k has been used in [5, 8, 11]. Whereas ω and γ are measures of weak noncompactness in the sense of the axiomatic definition given in [2] the function k fails to satisfy $\mathrm{k}(\operatorname{conv} H) = \mathrm{k}(H)$, see [12], that is one of the properties required in order to be a measure of weak noncompactness.

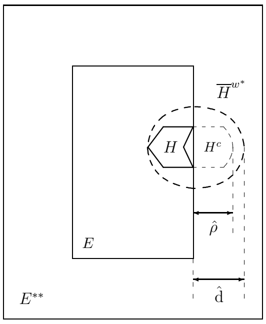

Figure 1

Nonetheless, k as well as γ and ω does satisfy the condition $\mathrm{k}(H) = 0$ if, and only if, H is relatively weakly compact in E. This fact for k is illustrated in the adjacent figure. Notice that for the bounded subset H of E the weak*-closure $\overline{H}^{w^*}$ in E^{**} is weak*-compact and therefore $\mathrm{k}(H) \overset{Figure\,1}{=:} \hat{d} = 0$ is equivalent to have $\overline{H}^{w^*} \subset E$ and thus equivalent to say that H is relatively weakly compact in E.

The paper is organized as follows.

In section 2, see Theorem 2.3, we prove that for any bounded subset in a Banach space E we have the inequalities

$$\mathrm{ck}(H) \leq \mathrm{k}(H) \leq \gamma(H) \leq 2\,\mathrm{ck}(H) \leq 2\,\mathrm{k}(H) \leq 2\omega(H).$$

By doing so we establish that ck, k and γ are equivalent; we provide a quantitative version of the angelicity of a Banach space for the weak topology. We study when $\mathrm{ck} = \mathrm{k}$ and we prove that this is the case for the class of Banach spaces with Corson property $\mathcal{C}$, Proposition 2.6. We also give an example for which $\mathrm{k}(H) = 2\,\mathrm{ck}(H)$, Example 2.5. Our results here can be viewed as a *quantitative* counterpart of the classical Eberlein-Smulyan's theorem about weak compactness in Banach spaces.

Section 3 is started with Lemma 3 that links the ε-interchanging of limits with a compact space and the ε-interchanging of limits with some dense subset of it. This is a common tool that is used to prove quantitative counterparts for γ of Gantmacher theorem about weak compactness of adjoint operators in Banach spaces, Theorem 3.1, and for the classical Grothendieck's characterization of weak compactness in spaces $C(K)$, Theorem 3.5. We complete this section commenting on the fact that for the De Blassi measure of weak noncompactness ω, Astala and Tylli proved in [1] that it is not possible to obtain a quantitative version of Gantmacher theorem similar to the one in Theorem 3.1: this provides another way of proving the fact commented in [1] that ω is not equivalent to the measure γ, see Corollary 3.4.

A bit of terminology: by letters $T, X, \dots$ we denote here sets or completely regular topological spaces, (Z, d) is a metric space. The space Z^X is equipped with the product topology τ_p. In Z^X we also consider the *standard supremum metric*, that abusively is also denoted by d, i.e.,

$$d(f, g) = \sup\{d(f(x), g(x)) : x \in X\}$$

for functions $f, g : X \to Z$. $C(X)$ is the space of continuous maps from X into the real line $\mathbb{R}$.

For A and B nonempty subsets of a metric space (Z, d), we consider the *usual distance* between A and B given by

$$d(A, B) = \inf\{d(a, b) : a \in A, b \in B\},$$

and the *Hausdorff non-symmetrized distance* from A to B defined by

$$\hat{d}(A, B) = \sup\{d(a, B) : a \in A\}.$$

In this paper $(E, \|\cdot\|)$ is a Banach space (E if $\|\cdot\|$ is tacitly assumed). B_E stands for the closed unit ball in E, E^* for the dual space and E^{**} for the bidual space; w is the weak topology of a Banach space and w^* is the weak* topology in a dual. We write $i : E \to E^{**}$ to denote the natural embedding of E into its bidual E^{**} and, as usual, most of the times we will not make any distinction between a given set $H \subset E$ and its image $H = i(H) \subset E^{**}$. In E^{**} we always consider the natural norm and its associated metric.

2. Measures of weak noncompactness in Banach spaces

Let H be a bounded subset of the Banach space E. If $\varphi \in H^{\mathbb{N}}$ is a sequence in H we write

$$\operatorname{clust}_{E^{**}}(\varphi) := \bigcap_{n \in \mathbb{N}} \overline{\{\varphi(m) : m > n\}}^{w^*}$$

to denote the set of cluster points of φ in (E^{**}, w^*). We also write $\overline{H}^{w^*}$ to denote the w^*-closure of H in E^{**}.

Definition 1. *Given a bounded subset H of a Banach space E we define:*

$$\omega(H) := \inf\{\varepsilon > 0 : H \subset K_\varepsilon + \varepsilon B_E \text{ and } K_\varepsilon \subset X \text{ is } w\text{-compact}\},$$

$$\gamma(H) := \sup\{|\lim_n \lim_m f_m(x_n) - \lim_m \lim_n f_m(x_n)| : (f_m) \subset B_{E^*}, (x_n) \subset H\},$$

assuming the involved limits exist,

$$\operatorname{ck}(H) := \sup_{\varphi \in H^{\mathbb{N}}} d(\operatorname{clust}_{E^{**}}(\varphi), E)$$

and

$$\operatorname{k}(H) := \hat{d}(\overline{H}^{w^*}, E) = \sup_{x^{**} \in \overline{H}^{w^*}} d(x^{**}, E),$$

where the distance d is the usual inf *distance for sets associated to the natural norm in E^{**}.*

Observe that for a bounded set $H \subset E$ we have

$$\operatorname{k}(H) := \inf\{\varepsilon > 0 : \overline{H}^{w^*} \subset E + \varepsilon B_{E^{**}}\}. \tag{2.1}$$

The notion below introduced in [5] was first considered by Grothendieck in [13], for $\varepsilon = 0$. For $\varepsilon \geq 0$, this concept has also been used, in the framework of Banach spaces, in [1, 8, 15] amongst others.

Definition 2. *Let (Z, d) be a metric space, X a set and $\varepsilon \geq 0$.*

(i) *We say that a sequence $(f_m)_m$ in Z^X ε-interchanges limits with a sequence $(x_n)_n$ in X if*

$$d(\lim_n \lim_m f_m(x_n), \lim_m \lim_n f_m(x_n)) \leq \varepsilon$$

whenever all limits involved do exist.

(ii) *We say that a subset H of Z^X ε-interchanges limits with a subset A of X, if each sequence in H ε-interchanges limits with each sequence in A. When $\varepsilon = 0$ we simply say that H interchanges limits with A.*

Observe that if H is a subset of a Banach space E, then $\gamma(H) \leq \varepsilon$ if, and only if $H\varepsilon$-interchanges limits with the dual ball B_{E^*}.

Our starting point for the results in this section are Propositions 2.1 and 2.2 that we quote below and Lemma 1 that we prove.

Proposition 2.1 ([5, Corollary 2.6] and [8, Proposition 8]). *Let E be a Banach space and let H be a bounded subset of E. The following properties hold:*

(i) *if H ε-interchanges limits with B_{X^*}, then $\mathrm{k}(H) \leq \varepsilon$,*
(ii) *if $\mathrm{k}(H) \leq \varepsilon$, then H 2ε-interchanges limits with B_{X^*}.*

Proposition 2.2 ([5, Proposition 5.2]). *Let (Z, d) be a compact metric space, K a set, and $H \subset Z^K$ a set which ε-interchanges limits with K. Then for any $f \in \overline{H}^{Z^K}$, there is a sequence $(f_n)_n$ in H such that*

$$\sup_{x \in K} d(g(x), f(x)) \leq \varepsilon$$

for any cluster point g of (f_n) in Z^K.

Lemma 1. *Let E be a Banach space and let H be a bounded subset of E. Then H $2\,\mathrm{ck}(H)$-interchanges limits with the dual ball B_{E^*}.*

Proof. Let $(f_m)_m$ be a sequence in B_{E^*}, $(x_n)_n$ a sequence in H and let us assume that both iterated limits

$$\lim_n \lim_m f_m(x_n), \lim_m \lim_n f_m(x_n)$$

exist in $\mathbb{R}$. If we fix $\alpha \in \mathbb{R}$ with $\alpha > \mathrm{ck}(H)$ the sequence $(x_n)_n$ has a w^*-cluster point $z \in E^{**}$ such that $d(z, E)) < \alpha$. Take and fix now $z' \in E$ such that

$$\|z - z'\| < \alpha. \tag{2.2}$$

Let us pick $f \in B_{E^*}$ a w^*-cluster point of $(f_m)_m$. Since z' and each x_n belongs to E, $f(z')$ and $f(x_n)$ are, respectively, cluster points in $\mathbb{R}$ of $f_m(z')$ and $f_m(x_n)$. Hence we can produce a subsequence $(f_{m_k})_k$ of $(f_m)_m$ such that $\lim_k f_{m_k}(z') = f(z')$. Thus we have that

$$\begin{aligned} &|\lim_k f_{m_k}(z) - f(z)| \leq \\ &\leq |\lim_k f_{m_k}(z) - \lim_k f_{m_k}(z')| + |f(z') - f(z)| \overset{(2.2)}{\leq} 2\alpha. \end{aligned} \tag{2.3}$$

We conclude that

$$\lim_n \lim_m f_m(x_n) = \lim_n f(x_n) = f(z)$$

and so

$$\begin{aligned} &|\lim_m \lim_n f_m(x_n) - \lim_n \lim_m f_m(x_n)| = \\ &= |\lim_m \lim_n f_m(x_n) - f(z)| = |\lim_k f_{m_k}(z) - f(z)| \overset{(2.3)}{\leq} 2\alpha. \end{aligned}$$

Since $\alpha > \mathrm{ck}(H)$ was arbitrary we obtain that H $2\,\mathrm{ck}(H)$-interchanges limits with B_{E^*}. $\square$

The above preparations lead naturally to the following result.

Theorem 2.3. *For any bounded subset H of a Banach space E we have:*

$$\mathrm{ck}(H) \leq \mathrm{k}(H) \leq \gamma(H) \leq 2\,\mathrm{ck}(H) \leq 2\,\mathrm{k}(H) \leq 2\omega(H), \tag{2.4}$$
$$\gamma(H) = \gamma(\mathrm{conv}(H)) \text{ and } \omega(H) = \omega(\mathrm{conv}(H)).$$

For any $x^{**} \in \overline{H}^{w^*}$, there is a sequence $(x_n)_n$ in H such that

$$\|x^{**} - y^{**}\| \leq \gamma(H) \tag{2.5}$$

for any cluster point y^{**} of $(x_n)_n$ in E^{**}.

Furthermore, H is weakly relatively compact in E if, and only if, one (equivalently all) of the numbers $\mathrm{ck}(H), \mathrm{k}(H), \gamma(H)$ and $\omega(H)$ is zero.

Proof. The inequality $\mathrm{ck}(H) \leq \mathrm{k}(H)$ straightforwardly follows from the definitions involved. The inequality $\mathrm{k}(H) \leq \gamma(H)$ is a consequence of statement (i) in Proposition 2.1. The inequality $\gamma(H) \leq 2\,\mathrm{ck}(H)$ follows from Lemma 1.

The approximation (2.5) straightforwardly follows from Proposition 2.2 after the convenient identification of $(\overline{H}^{w^*}, w^*)$ as a subspace of $([-M, M]^{B_{E^*}}, \tau_p)$ where M is a bound for H.

On the other hand $\gamma(H) = \gamma(\mathrm{conv}(H))$ has been established in [8, Theorem 13] and [5, Theorem 3.3]. The equality $\omega(H) = \omega(\mathrm{conv}(H))$ follows from the very definition of ω using that the validity of Krein-Smulyan theorem stating that the closed convex hull of weakly compact sets in Banach spaces are weakly compact.

A well known result of Grothendieck, [7, Lemma 2, p. 227] states that $\omega(H) = 0$ if, and only if, H is relatively weakly compact in E. Observe that as a consequence of (2.4) one of the numbers $\mathrm{ck}(H)$, $\mathrm{k}(H)$ $\gamma(H)$ is zero if, and only if, all of them are zero. Clearly, $\mathrm{k}(H) = 0$ if, and only if, $\overline{H}^{w^*} \subset E$ that is equivalent to the fact that H is relatively weakly compact.

To finish we prove the very last inequality in (2.4). Take $\varepsilon > 0$ and a weakly compact set $K_\varepsilon \subset E$ such that $H \subset K_\varepsilon + \varepsilon B_E$. We have that

$$\overline{H}^{w^*} \subset K_\varepsilon + \varepsilon B_{E^{**}} \subset E + \varepsilon B_{E^{**}}.$$

If we use (2.1) we obtain $k(H) \leq \omega(H)$ and the proof is over. □

We refer the interested reader to [15] where measures of weak noncompactness are defined: all of the conditions there are fulfilled by γ and ω and most of them by ck and k. As a consequence of the above ck, k, γ are *equivalent* while ω is *not equivalent* to the other ones, see Corollary 3.4.

A topological space T is said to be *angelic* if, whenever H is a relatively countably compact subset of T, its closure $\overline{H}$ is compact and each element of $\overline{H}$ is a limit of a sequence in H. Our references for angelic spaces are [10] and [17]. Theorem 2.3 above is the quantitative version of the angelicity of a Banach space endowed with its weak topology, Eberlein-Smulyan's theorem.

Corollary 2.4. *If E is a Banach space then (E, w) is angelic.*

Proof. Let H be a w-relatively countably compact subset of E. By the very definition every sequence in H has a w-cluster point in E and therefore $\mathrm{ck}(H) = 0$. Then by Theorem 2.3 we have H is w-relatively compact in E. On the other hand, let us pick $x \in \overline{H}^{w}$. Note that inequality (2.4) implies that $\gamma(H) = 0$ and thus if we use (2.5) we obtain the existence of a sequence $(x_n)_n$ in H such that every

w-cluster point $y \in E$ of $(x_n)_n$ satisfies that $0 \le \|y - x\| \le \gamma(H) = 0$. Since H is w-relatively compact and $(x_n)_n$ in H and x is the unique w-cluster point of $(x_n)_n$ we conclude that $(x_n)_n$ weakly converges to x and the proof is over. $\square$

Talking about ck and k, it is pretty easy to prove that if E^* is separable for the norm, then for every bounded set $H \subset E$ we have $\mathrm{ck}(H) = \mathrm{k}(H)$. Keeping this in mind it is easy to produce an example showing that constant 2 for the inequality $\gamma(H) \le 2\,\mathrm{ck}(H)$ it is truly needed: indeed, take $E = c$ the space of convergent real sequences and $H := B_c$ its unit ball. On the one hand $\mathrm{ck}(B_c) = \mathrm{k}(B_c)$ is equal to 1 after Riesz lemma and on the other hand considering elements of the type $(1, \dots, 1, -1, \dots, -1, \dots)$ and the n-th projections $\pi_n : c \to \mathbb{R}$ one easily computes that $\gamma(B_c) = 2$, see [15, p. 93].

Now we give a more involved example showing that even for the inequality $\mathrm{k}(H) \le 2\,\mathrm{ck}(H)$ the constant 2 is needed.

Example 2.5. The following example has been communicated to us by Prof. Marciszewski. Consider $[0, \omega_1]$ the compact set of all the ordinals smaller or equal to the first non countable ordinal ω_1. Put

$$K = (\{-1, 1\} \times [0, \omega_1])/R$$

where R is the relation defined as xRy if, and only if

$$x = y \text{ or } x, y \in \{(-1, \omega_1), (1, \omega_1)\}.$$

Clearly K is a compact set. For $\alpha \prec \omega_1$ define $f_\alpha : K \to \mathbb{R}$ as

$$f_\alpha(i, \gamma) = \begin{cases} 0 & \text{if } \gamma \succ \alpha, \\ i & \text{if } \gamma \preceq \alpha \end{cases}$$

and put $H = \{f_\alpha : \alpha \prec \omega_1\} \subset C(K)$. Since H is uniformly bounded and K is scattered, the w^*-topology in $\overline{H}^{w^*} \subset C(K)^{**} = \ell^\infty(K)$ coincides with the product topology of $\mathbb{R}^K$. If $(f_{\alpha_n})_n$ is a sequence in H and $\alpha := \sup\{\alpha_n : n \in \mathbb{N}\}$ then $\alpha \prec \omega_1$ and $f_{\alpha_n}(i, \beta) = 0$ for all $n \in \mathbb{N}$ and $\beta \succ \alpha$. So for every $\beta \succ \alpha$ we have that $g(i, \beta) = 0$ for each cluster point g of $(f_{\alpha_n})_n$. If we define $h : K \to \mathbb{R}$ as $h(i, \beta) = 0$ if $\beta \succ \alpha$ and $h(i, \beta) = i/2$ if $\beta \preceq \alpha$ then $h \in C(K)$ and $d(h, g) \le 1/2$ for each cluster point g of $(f_{\alpha_n})_n$. Thus we conclude that $\mathrm{ck}(H) \le 1/2$. On the other hand, the function $h' : K \to \mathbb{R}$ defined as $h'(i, \beta) = 0$ if $\beta = \omega_1$ and $h'(i, \beta) = i$ if $\beta \neq \omega_1$ belongs to $\overline{H}^{\mathbb{R}^K}$ and clearly $d(h', C(K)) = 1$. Then

$$\mathrm{k}(H) = \hat{d}(\overline{H}^{w^*}, C(K)) \ge d(h', C(K)) \ge 1 \ge 2\,\mathrm{ck}(H)$$

and therefore by Theorem 2.3 $d(\overline{H}^{w^*}, C(K)) = 2\,\mathrm{ck}(H)$. $\square$

We will devote the rest of the section to prove that the equality $\mathrm{ck} = \mathrm{k}$ holds for a pretty wide class of Banach spaces E enjoying Corson property $\mathcal{C}$. To do so we isolate first the following easy lemma that is inspired by the proof of [8, Proposition 14].

Lemma 2. *If $x^{**} \in E^{**} \setminus E$ and $b \in \mathbb{R}$ satisfies $d(x^{**}, E) > b > 0$, then*

$$0 \in \overline{\{x^* \in B_{E^*} : x^{**}(x^*) > b\}}^{w^*}.$$

Proof. We simply prove that for each $\varepsilon > 0$ and finitely many $x_1, x_2, \dots, x_n \in E$ the w^*-neighborhood of the origin in E^*

$$V(0, x_1, x_2, \dots, x_n, \varepsilon) := \{y^* \in E^* : \sup_{1\le i\le n} |y^*(x_i)| < \varepsilon\}$$

intersects the set $S(x^{**}, b) := \{x^* \in B_{E^*} : x^{**}(x^*) > b\}$. Hahn-Banach's theorem provides us with a functional $\phi \in E^{***}$ such that $\phi(x) = 0$ for every $x \in E$, $\|\phi\| = 1$ and $\phi(x^{**}) = d(x^{**}, E)$, [6, Corollary 6.8]. We can and do assume that $b < b + \varepsilon < d(x^{**}, X)$. We use Goldstine's theorem for $B_{E^*} \subset B_{E^{***}}$ to find an element x^* in B_{E^*} such that

$$|\phi(x_i) - x^*(x_i)| = |x^*(x_i)| < \varepsilon, i = 1, 2, \dots, n, \tag{2.6}$$

and

$$|\phi(x^{**}) - x^{**}(x^*)| < \varepsilon. \tag{2.7}$$

The inequalities (2.6) imply that $x^*, -x^* \in V(0, x_1, x_2, \dots, x_n, \varepsilon)$. On the other hand inequality (2.7) implies that

$$\begin{aligned}|x^{**}(x^*)| = |(x^{**}(x^*) - \phi(x^{**})) + \phi(x^{**})| &\ge \big||x^{**}(x^*) - \phi(x^{**})| - |\phi(x^{**})|\big| \\ &= |\phi(x^{**})| - |x^{**}(x^*) - \phi(x^{**})| > b + \varepsilon - \varepsilon = b.\end{aligned}$$

All things considered, either x^* or $-x^*$ belongs to

$$V(0, x_1, x_2, \dots, x_n, \varepsilon) \cap S(x^{**}, b)$$

and the proof is over. □

Recall that a Banach space E is said to have Corson property $\mathcal{C}$ if each collection of closed convex subsets of E with empty intersection has a countable subcollection with empty intersection. If (E, w) is Lindelöf, then E has property $\mathcal{C}$. There are Banach spaces with Corson property $\mathcal{C}$ which are not weakly Lindelöf, [18, p. 146]. It is shown in [18] that the Banach space E has the property $\mathcal{C}$ if and only if, whenever $A \subset E^*$ and $x^* \in \overline{A}^{w^*}$, there is a countable subset C of A such that $x^* \in \overline{\operatorname{conv} C}^{w^*}$. In particular Banach spaces with w^* angelic dual unit balls have Corson property $\mathcal{C}$.

Proposition 2.6. *If E is a Banach space with Corson property $\mathcal{C}$, then for every bounded set $H \subset E$ we have* $\operatorname{ck}(H) = \operatorname{k}(H)$.

Proof. We already know that $\operatorname{ck}(H) \le \operatorname{k}(H)$, Theorem 2.3. Therefore if $\operatorname{k}(H) = 0$ the equality holds. Otherwise, we prove that for every $0 < b < \operatorname{k}(H)$ we have $b \le \operatorname{ck}(H)$ that clearly implies $\operatorname{ck}(H) = \operatorname{k}(H)$. For such a b we take $x^{**} \in \overline{H}^{w^*} \setminus E$ with $d(x^{**}, E) > b$. Lemma 2 tells us that if we write

$$S(x^{**}, b) := \{x^* \in B_{E^*} : x^{**}(x^*) > b\}$$

then $0 \in \overline{S(x^{**}, b)}^{w^*}$. Now, property $\mathcal{C}$ of E applies to provide us with a countable subset C of $S(x^{**}, b)$ such that $0 \in \overline{\operatorname{conv} C}^{w^*}$. Since $S(x^{**}, b)$ is convex, there is a countable set D of $S(x^{**}, b)$ such that $0 \in \overline{D}^{w^*}$. Since D is countable, $\overline{H}^{w^*}$ is pseudo-metrizable in the topology of pointwise convergence on D. So one can choose a sequence $(h_n)_n$ in H that converges to x^{**} pointwise on D. Therefore, if h^{**} is any w^*-cluster point of $(h_n)_n$, then $h^{**}|_D = x^{**}|_D$. In particular, we have that

$$h^{**}(x^*) = x^{**}(x^*) > b,$$

for each $x^* \in D$. On the other hand, since $0 \in \overline{D}^{w^*}$, for fixed arbitrary $h \in E$ and $\varepsilon > 0$ there is some $x^* \in D$ such that $|x^*(h)| < \varepsilon$. Consequently

$$\|h^{**} - h\| \geq h^{**}(x^*) - x^*(h) > b - \varepsilon.$$

Since ε and h are arbitrary we conclude that $d(h^{**}, E) \geq b$ for every w^*-cluster point h^{**} of $\varphi = (h_n)_n$. We conclude that

$$\mathrm{ck}(H) \geq d(\mathrm{clust}_{E^{**}}(\varphi), E) \geq b$$

and the proof is over. □

A different proof of Proposition 2.6 can be given for the particular case of Banach spaces E with angelic dual unit ball (B_{E^*}, w^*) that we sketch briefly: in this case we argue by contradiction. We assume that there is a bounded set $H \subset E$ such that $\mathrm{ck}(H) < b < \mathrm{k}(H)$. Then we proceed as we did in the proof of Proposition 2.6 taking $x^{**} \in \overline{H}^{w^*} \setminus E$ with $d(x^{**}, E) > b$ and $0 \in \overline{S(x^{**}, b)}^{w^*}$. The angelicity of (B_{E^*}, w^*) provides us with a sequence $(x_n^*)_n$ in $S(x^{**}, b)$ with $w^* - \lim_n x_n^* = 0$. If we define now the linear operator

$$\begin{aligned} T : E &\longrightarrow c_0 \\ x &\rightsquigarrow (x_n^*(x))_n \end{aligned}$$

then $\|T\| \leq 1$ and one readily computes that $\mathrm{ck}(T(H)) \leq \mathrm{ck}(H)$. Following up the proof of Theorem 3 in [11] one concludes that $d(T^{**}(x^{**}), c_0) \geq b$ that leads to

$$k(T(H)) \geq d(T^{**}(x^{**}), c_0) \geq b > ck(H) \geq ck(T(H)),$$

that contradicts that $\mathrm{ck} = \mathrm{k}$ in c_0 because $c_0^* = \ell_1$ is separable.

We gratefully acknowledge the comments of Professor V. Kadets that upon the reading of a preliminary version of this paper, where Proposition 2.6 was proved for Banach spaces with angelic dual unit ball, came up with some of the ideas we have presented now that works for the more general Banach spaces with Corson property $\mathcal{C}$.

Observe that $\mathrm{ck}(H) = \mathrm{k}(H)$ implies that $\hat{d}(\overline{H}^{w^*}, E) = \hat{d}(H^c, E)$ where

$$H^c := \bigcup_{\varphi \in H^{\mathbb{N}}} \mathrm{clust}_{E^{**}}(\varphi),$$

although it might happen that $H^c \subsetneq \overline{H}^{w^*}$. Indeed, if Γ is a non countable set then $c_0(\Gamma)$ is weakly compactly generated, hence weakly Lindelöf and in particular it has Corson property $\mathcal{C}$. Therefore $\mathrm{ck} = \mathrm{k}$ in $c_0(\Gamma)$. On the other hand, the unit ball $H := B_{c_0(\Gamma)}$ and H^c are made up of functions defined of Γ with countable support and consequently $B_{\ell^\infty(\Gamma)} = \overline{H}^{w^*}$ contains properly H^c: we have tried to illustrate H^c in the Figure 1 and where have written $\hat{\rho} = \hat{d}(H^c, E)$.

3. Quantitative versions of Gantmacher and Grothendieck theorems

The Hausdorff measure of norm noncompactness is defined for bounded sets H of Banach spaces E as

$$\mathrm{h}(H) := \inf\{\varepsilon > 0 : H \subset K_\varepsilon + \varepsilon B_E \text{ and } K_\varepsilon \subset X \text{ is finite}\}.$$

A theorem of Schauder states that a continuous linear operator $T : E \to F$ is compact if, and only if, its adjoint operator $T^* : F^* \to E^*$ is compact. A quantitative strengthening of Schauder's result was proved by Goldenstein and Marcus (*cf.* [1, p. 367]) who established the inequalities

$$\frac{1}{2}\,\mathrm{h}(T(B_E)) \leq \mathrm{h}(T^*(B_{F^*})) \leq 2\,\mathrm{h}(T(B_E)). \tag{3.1}$$

For weak topologies Gantmacher established that the operator T is weakly compact if, and only if, T^* is weakly compact. Nonetheless, the corresponding quantitative version to (3.1) where h is replaced by ω fails for general Banach spaces, see Remark 3.3. On the positive side we prove in Theorem 3.1 a quantitative version of Gantmacher result for γ. In order to do this we need first the lemma below that can be obtained combining Propositions 2.2 and 2.4 in [5]: we prefer to include a selfcontained straightforward proof for the lemma though.

Lemma 3. *Let K be a compact topological space, D a dense subset of K, H a uniformly bounded subset of $C(K)$ and $\varepsilon > 0$. If H ε-interchanges limits with D, then H 2ε-interchanges limits with K.*

Proof. Fix $\delta > \varepsilon$. We first prove the claim below:

CLAIM: If $f \in \overline{H}^{\mathbb{R}^K}$, then for every $y \in K$ there exist a neighborhood V of y in K such that

$$\sup_{d\in V\cap D} |f(d) - f(y)| \leq \delta. \tag{3.2}$$

We prove the claim by contradiction: we assume that $\sup_{d\in U\cap D} |f(d)-f(y)| > \delta$ for each neighborhood U of y and we will contradict that H ε-interchanges limits with D. Indeed, let us write $d_0 = y$. Since $f \in \overline{H}^{\mathbb{R}^K}$, there exist $g_1 \in H$ such that $|f(d_0) - g_1(d_0)| \leq 1$. Since g_1 is continuous there exist a neighborhood U of y such that $|g_1(d_0) - g_1(d)| \leq 1$ for all d in U. By assumption, there is $d_1 \in U \cap D$ such that $|f(d_1) - f(d_0)| > \delta$. Proceeding by recurrence we produce sequences $(d_n)_n$ in D and $(g_n)_n$ in H such that for every $n \in \mathbb{N}$ we have

$$|g_n(d_i) - f(d_i)| \leq \frac{1}{n} \quad i = 0, 1, \ldots, n-1 \tag{3.3}$$

$$|g_j(d_n) - g_j(d_0)| \leq \frac{1}{n} \quad j = 1, \ldots, n \tag{3.4}$$

$$|f(d_n) - f(d_0)| > \delta. \tag{3.5}$$

Choosing a subsequence we can assume that $(f(d_n))_n$ converges in $\mathbb{R}$. Then we have

$$\lim_n \lim_m g_m(d_n) \stackrel{(3.3)}{=} \lim_n f(d_n),$$

$$\lim_m \lim_n g_m(d_n) \stackrel{(3.4)}{=} \lim_m g_m(d_0) \stackrel{(3.3)}{=} f(d_0) = f(y),$$

so

$$|\lim_n \lim_m g_m(d_n) - \lim_m \lim_n g_m(d_n)| = |\lim_n f(d_n) - f(y)| \stackrel{(3.5)}{\geq} \delta,$$

that contradicts that H ε-interchanges limits with D and finishes the proof of the claim.

We finish now the proof of the lemma. Take sequences $(x_n)_n$ in K and $(f_m)_m$ in H for which the double limits $\lim_n \lim_m f_m(x_n)$ and $\lim_m \lim_n f_m(x_n)$ exist.

If we take $f \in \overline{H}^{\mathbb{R}^K}$ and $x \in K$, cluster points of $(f_m)_m$ in $\mathbb{R}^K$ and $(x_n)_n$ in K, respectively, then we have

$$\lim_m \lim_n f_m(x_n) = \lim_m f_m(x) = f(x)$$

and

$$\lim_n \lim_m f_m(x_n) = \lim_n f(x_n).$$

Consequently we obtain that

$$|\lim_n \lim_m f_m(x_n) - \lim_m \lim_n f_m(x_n)| = |\lim_n f(x_n) - f(x)| = L.$$

By the claim there is a neighborhood U of x such that $\sup_{d\in U\cap D} |f(x)-f(d)| \le \delta$. For every n in $\mathbb{N}$, there exist $k > n$ such that $x_k \in U$. Now the claim applies again to provide us with a neighborhood V of x_k contained in U such that $\sup_{d\in V\cap D} |f(x_k) - f(d)| \le \delta$. If we pick $d_k \in V \cap D$, we have that

$$|f(x_k) - f(x)| \le |f(x_k) - f(d_k)| + |f(d_k) - f(x)| \le 2\delta.$$

Thus $L \le 2\delta$ and since we can repeat this argument for any arbitrary $\delta > \varepsilon$, we conclude that H 2ε-interchanges limits with K. □

Theorem 3.1. *Let E and F be Banach spaces, $T : E \to F$ an operator and $T^* : F^* \to E^*$ its adjoint. Then*

$$\gamma(T(B_E)) \le \gamma(T^*(B_{F^*})) \le 2\gamma(T(B_E)).$$

Proof. If we take sequences $(x_n)_n$ in B_E and $(y_m^*)_m$ in B_{F^*}, the very definition of T^* implies that

$$\begin{aligned} \lim_n \lim_m y_m^*(T(x_n)) &= \lim_n \lim_m T^*(y_m^*)(x_n), \\ \lim_m \lim_n y_m^*(T(x_n)) &= \lim_m \lim_n T^*(y_m^*)(x_n) \end{aligned} \tag{3.6}$$

whenever the limits in the left hand sides (or the right hand sides) do exist. Hence, if $(x_n)_n$ and $(y_m^*)_m$ are as above assuming that the limits on the left hand side of (3.6) exist then

$$|\lim_n \lim_m y_m^*(T(x_n)) - \lim_m \lim_n y_m^*(T(x_n))| \le \gamma(T^*(B_{F^*})).$$

Consequently we obtain that $\gamma(T(B_E)) \le \gamma(T^*(B_{F^*}))$.

The other way around, if $(x_n)_n$ and $(y_m^*)_m$ are as above assuming that the limits on the right hand side of (3.6) exist then

$$|\lim_n \lim_m T^*(y_m^*)(x_n) - \lim_m \lim_n T^*(y_m^*)(x_n)| \le \gamma(T(B_E)).$$

In other words, we get that $T^*(B_{F^*}) \subset C(B_{E^{**}}, w^*)$ $\gamma(T(B_E))$-interchanges limits with $B_E \subset B_{E^{**}}$. Since B_E is w^*-dense in $B_{E^{**}}$ we can apply Lemma 3 to obtain $\gamma(T^*(B_{F^*})) \le 2\gamma(T(B_E))$. □

Corollary 3.2 (Gantmacher)**.** *Let E and F be Banach spaces, $T : E \to F$ an operator and $T^* : F^* \to E^*$ its adjoint. T is weakly compact if, and only if, T^* is weakly compact.*

Proof. Theorems 3.1 and 2.3 apply to conclude that $\gamma(T(B_E)) = 0$ (*i.e.* $T(B_E)$ is relatively weakly compact) if, and only if, $\gamma(T^*(B_{F^*})) = 0$ (*i.e.* $T^*(B_{F^*})$ is relatively weakly compact). □

Remark 3.3. Astala and Tylli constructed in [1, Theorem 4] a separable Banach space E and a sequence $(T_n)_n$ of operators $T_n : E \to c_0$ such that

$$\omega(T_n^*(B_{\ell^1})) = 1 \qquad \text{and} \qquad \omega(T_n^{**}(B_E^{**})) \leq w(T_n(B_E)) \leq \frac{1}{n}.$$

Note that this example says, in particular, that there are no constants $m, M > 0$ such that for any bounded operator $T : E \to F$ we have

$$m\omega(T(B_E)) \leq \omega(T^*(B_{F^*})) \leq M\omega(T(B_E)).$$

Corollary 3.4. *γ and ω are not equivalent measures of weak noncompactness, namely there is no $N > 0$ such that for any Banach space and any bounded set $H \subset E$ we have*

$$\omega(H) \leq N\gamma(H). \tag{3.7}$$

Proof. If we assume that there is N satisfying (3.7), then inequality (2.4) allows us to complete inequality (3.7) as

$$\frac{1}{2}\gamma(H) \leq \omega(H) \leq N\gamma(H),$$

for any bounded subset H of any arbitrary Banach space E. Theorem 3.1 applies to conclude that for any bounded operator between arbitrary Banach spaces $T : E \to F$ we have to have

$$\frac{1}{2N}\omega(T(B_E)) \leq \omega(T^*(B_{F^*})) \leq 4N\omega(T(B_E))$$

that contradicts the example in Remark 3.3. □

We have to stress that the fact that γ and ω are not equivalent has been noted in [1, Corollary 5 and p. 372]: Astala and Tylli proved in their Corollary 5 that there exist a separable Banach space E, a linear isometry $J : E \to \ell^\infty$ and a sequence $(B_n)_n$ of bounded sets of E with $\omega(B_n) = 1$ and $\omega(JB_n) \leq \frac{1}{n}$ for each $n \in \mathbb{N}$. But $\gamma(IB) = \gamma(B)$ for all linear isometries I so γ and ω are not equivalent.

To finish, we give another application of the techniques we have developed here: we prove a quantitative strengthening of Grothendieck's classical characterization of weakly compact sets in spaces $C(K)$. If $H \subset C(K)$ we define

$$\gamma_K(H) := \sup\{|\lim_n \lim_m f_m(x_n) - \lim_m \lim_n f_m(x_n)| : (f_m) \subset H, (x_n) \subset K\}.$$

Theorem 3.5. *Let K be a compact space and let H be a uniformly bounded subset of $C(K)$. Then we have*

$$\gamma_K(H) \leq \gamma(H) \leq 2\gamma_K(H).$$

Proof. The inequality $\gamma_K(H) \leq \gamma(H)$ is clear. Let us prove the second inequality: fix $M > 0$ a uniform bound for H. For every $x \in K$ let us write $\delta_x : C(K) \to \mathbb{R}$ to denote the Dirac measure at x and let us define $D := \{\pm\delta_x : x \in K\}$. If we consider $D|_H \subset [-M, M]^H$, then $D|_H$ $\gamma_K(H)$-interchanges limits with H, therefore we can apply [5, Theorem 3.3] to obtain that $\mathrm{conv}(D)|_H$ $\gamma_K(H)$-interchanges limits with H. In other words, H $\gamma_K(H)$-interchanges with $\mathrm{conv}(D) \subset B_{C(K)^*}$. Since $\mathrm{conv}(D)$ is w^*-dense in $B_{C(K)^*}$ we can apply Lemma 3 to obtain that H $2\gamma_K(H)$-interchanges with $B_{C(K)^*}$, *i.e.*, $\gamma(H) \leq 2\gamma_K(H)$. □

Since a bounded subset H of a Banach space is τ_p-relatively compact if, and only if $\gamma_K(H) = 0$ (see [10, p.12] or [5, Corollary 2.5]), we get the following corollary.

Corollary 3.6. *Let K be a compact space and let H be a uniformly bounded subset of $C(K)$, then H is τ_p-relatively compact if, and only if, H is w-relatively compact.*

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